On Estimation of Reliability Functions using Record values from Proportional Hazard Rate Model

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Abstract

Two measures of reliability functions, namely $R(t) = P(X > t)$ and $P = P(X < Y)$ have been studied based on record values from proportional hazard rate model (PHR) model. For estimation of P, we generalize the results of Basirat et al. (2016) when X and Y belong to different family of distributions from PHR model. Uniformly minimum variance unbiased estimator (UMVUE), maximum likelihood estimator (MLE) and Bayes estimator (BS) are obtained for the powers of the parameter and reliability functions. Simulation studies and a real data example have been presented for illustrative purposes. Asymptotic and exact confidence intervals of the parameters and reliability functions are constructed. Testing procedures are also developed for various hypotheses.

Keywords: proportional hazard rate model, records, maximum likelihood estimator, uniformly minimum variance unbiased estimator, Bayes estimator, confidence intervals.

1. Introduction

The most popularly used parameter in life testing analysis and reliability engineering is the reliability function. This function is defined as the probability of a component operating for a certain amount of time without failure. As such, the reliability function is a function of time t and is mathematically defined as $R(t) = P(X > t)$ where the random variable X denotes the lifetime of the component. Every physical component possesses an inherent strength on the basis of which it survives. These components are subjected to a certain level of failure inducing stress. If higher level of stress is enforced on the components, then they are unable to sustain it and eventually break down. The fluctuations in strength and stress can be modelled to a statistical distribution and a natural scatter arises in the two variables when their respective distributions interact. Hence, the reliability component may be determined analytically when the probability density functions of both the stress and strength variables are well defined. Therefore, we define $P = P(X < Y)$ which represents the reliability of a component of random strength Y subject to random stress X. In the stress-strength setup, a system fails if and only if at any given time, the applied stress exceeds the strength and was considered by Birnbaum (1956). Stress-strength models have great utility in

the fields of genetics, psychology, engineering and so on. Interested readers may refer to Kotz et al. (2003) for further information about stress-strength parameter. In statistical literature, inferences have been drawn on P under several assumptions on X and Y. Rezaei, Tahmasbi, and Mahmoodi (2010) estimated P when X and Y are independent random variables from generalized Pareto distribution with common scale parameter and different shape parameters. Kundu and Gupta (2005), Raqab, Madi, and Kundu (2008), Jiang and Wong (2008), Huang, Mi, and Wang (2012), Soliman, Abd-Ellah, Abou-Elheggag, and Ahmed (2013) are among a few to draw inferences on P based on complete sample. Some inferential procedures on P based on progressively Type II censored samples have been given by Asgharzadeh, Valiollahi, and Raqab (2011), Lio and Tsai (2012), Basirat, Baratpour, and Ahmadi (2013) and so on. Based on record values, one may refer to the work of Baklizi (2008a) who compared the likelihood and Bayesian estimation of stress-strength parameter using lower records from generalized exponential model. Baklizi (2008b) estimated P based on records from one and two parameter exponential distributions. Chaturvedi and Malhotra (2016, 2017) developed inferential procedures for $R(t)$ and P of a family of lifetime distributions and three parameters Burr XII distribution respectively.

Chandler (1952) introduced the concept of record values. Often, statistical data are more readily available in terms of records. In fact, data that are not records may not be available at all. The hottest day ever, the longest winning streak in professional basketball, the lowest stock market figure, are a case in point. Such type of data can be analyzed as record values from a sequence of independent and identically distributed random variables. In statistics, an upper (lower) record value is the largest (smallest) value obtained from a sequence of random variables. The theory of record values is closely related to the theory of extreme order statistics. Based on records, inferential procedures for the parameters of different distributions have been developed by Glick (1978), Arnold, Balakrishnan, and Nagaraja (1998), Ahmadi and Arghami (2003a,b), and others.

Proportional hazard rate (PHR) models pertain to a family of survival models proposed by Cox (1972) and have been primarily used in clinical testing analysis to model the effect of secondary variables on survival. Basirat *et al.* (2013) gave statistical inferences on P of PHR model based on progressive Type II censored data. Later, Basirat, Baratpour, and Ahmadi (2016) estimated P based on record values from PHR model and obtained its MLE, UMVUE and Bayes estimator. Their estimation procedures assumed that X and Y belong to the same family of distribution but have different shape parameters. In this paper, for estimation of P, we have extended and generalized the results of Basirat *et al.* (2016) when X and Y belong to different family of distributions. The results of Basirat *et al.* (2016) have been shown to be particular cases of our result. In our approach, we first obtain the estimators of the powers of the parameter of PHR model and using these estimators we estimate the probability density function (*pdf*). The estimated *pdf* is subsequently used to estimate $R(t)$ and P. This approach is adopted for classical as well as Bayesian estimation procedures. Thus, we have established a relationship between different estimation problems. In order to obtain UMVUE of P, Basirat *et al.* (2016) performed Rao-Blackwellisation which is avoided in this paper. Also, we do not require the derivation of conditional distribution of unbiased estimator of $R(t)$ and P while deriving their UMVUE.

The rest of the paper has been organised as follows. In Section 2, we define the PHR model and mention some of its particular cases. We also formally define record values. In Section 3, we derive MLE of the powers of the parameter and reliability functions $R(t)$ and P. Similarly in Sections 4 and 5, we derive UMVUE and Bayes estimators respectively of the powers of the parameter and reliability functions $R(t)$ and P. In Section 6, asymptotic and exact confidence intervals (CI) for the parameter and reliability functions are constructed. In Section 7, testing procedures are developed for the parametric functions of the distribution. In Section 8 we present a real data example for illustrative purposes and finally in Section 9 we numerically analyse the behaviour of various estimators. The paper ends with a brief discussion on our results.

2. The proportional hazard rate model and record values

Let X be a random variable with support S_x from PHR model with survival function defined as

$$\overline{F}(x;\alpha) = \left(\overline{F}_o(x)\right)^{\alpha} ; x \in S_x, \quad \alpha > 0 \tag{1}$$

We denote the cumulative distribution function (*cdf*) of the baseline distribution by $F_o(x)$ and the *cdf* of PHR model by $F(x;\alpha)$. Using these notations in (1) we define $\overline{F}(x;\alpha) = 1 - F(x;\alpha)$ where α is the shape parameter of the PHR model and $\overline{F}_o(x) = 1 - F_o(x)$. Thus, the *pdf* of X is $f(x;\alpha) = \alpha f_o(x)\left(\overline{F}_o(x)\right)^{\alpha-1}$ where $f_o(x)$ is the derivative of $F_o(x)$ with respect to x. The PHR model has some of its specific cases as exponential distribution, Weibull distribution, Rayleigh distribution, Burr distribution, Pareto distribution and so on.

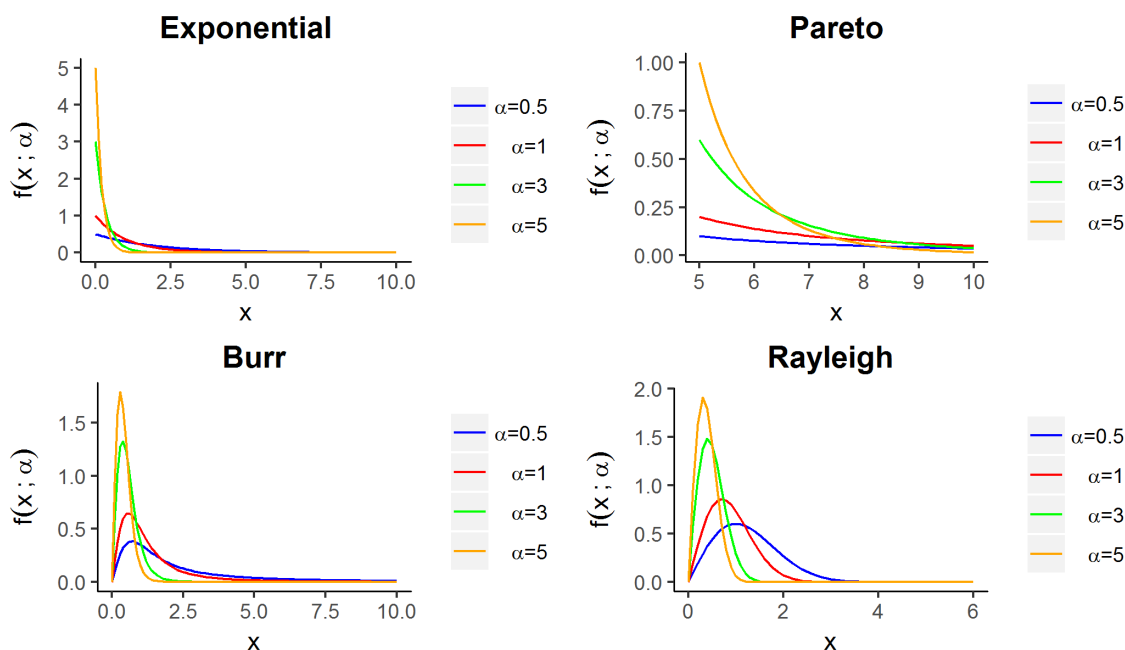The following figure shows the density curve of some of the distributions from PHR model.

Figure 1: The probability density curve of some of the distributions belonging to proportional hazard rate model for different values of the parameter α.

Let $X_1, X_2, \ldots$ be an infinite sequence of independent and identically distributed random variables from the PHR model with *df* $f(x;\alpha)$. An observation X_j will be called an upper record value (or simply a record) if its value exceeds than all of the previous observations. Thus X_j is a record if $X_j > X_i$ for every $i < j$. The record time sequence $\{T_n, n = 0\}$ is defined as

$$\begin{cases} T_0 = 1 ; & \text{with probability 1} \\ T_n = \min\{j : X_j > X_{T_{n-1}}\}; n = 1 \end{cases}$$

and the record value sequence $\{R_n\}$ is then given by

$$R_n = X_{T_n}; \quad n = 0, 1, 2, \ldots$$

The likelihood function of α given the first $n + 1$ upper record values $R_0, R_1, R_2, \ldots, R_n$ is

$$L(\alpha|R_0, R_1, R_2, \ldots, R_n) = f(R_n;\alpha) \prod_{i=0}^{n-1} \frac{f(R_i;\alpha)}{1 - F(R_i;\alpha)}$$

It is easy to see that

$$L\left(\alpha | R_0, R_1, R_2, \ldots, R_n\right) = \alpha^{n+1} exp\left(-\alpha U(R_n)\right) \prod_{i=0}^{n} \frac{f_o(R_i)}{1 - F_o(R_i)} \tag{2}$$

where $U\left(x\right) = \log\left(\frac{1}{1-F_o(x)}\right)$. It follows from (2) and factorisation theorem [see Rohtagi and Saleh (2012, p.361)] that $U(R_n)$ is a sufficient statistic for α and follows gamma distribution with shape parameter $n + 1$ and rate parameter α. Since the distribution of $U(R_n)$ belongs to exponential family, it is also complete [see Rohtagi and Saleh (2012, p.367)]. In order to estimate $P = P(X < Y)$, let Y be another random variable independent of X from a different family of PHR model with *pdf* $g\left(y; \beta\right) = \beta g_o(y)\left(\overline{G}_o\left(y\right)\right)^{\beta-1}$. Let $R_o^*, R_1^*, \ldots, R_m^*$ be the $m+1$ record values from the distribution of Y. For simplicity, we define $V\left(y\right) = \log\left(\frac{1}{1-G_o(y)}\right)$. Then following the same arguments as above, $V(R_m^*)$ is a complete and sufficient statistic for β and follows Gamma$(m + 1, \beta)$ distribution.

3. MLE

The fundamental principle of maximum likelihood estimation is that it assumes the sample is a representative of the population and estimates the population parameter in such a way that its value maximises the *pdf* of the distribution under study. The MLE of the parameter need not be an unbiased estimator. The log-likelihood function from (2) gives the MLE of α^q where $q \in (-\infty, \infty)$ and $q \neq 0$, as

$$\widehat{\alpha^q} = \left(\frac{n+1}{U(R_n)}\right)^q \tag{3}$$

By invariance property of MLE, the MLE of $R\left(t\right)$ is

$$\hat{R}\left(t\right) = exp\left(-(n+1)\frac{U(t)}{U(R_n)}\right) \tag{4}$$

Using the fact that $\hat{f}\left(x; \alpha\right) = -\frac{d}{dx}\hat{R}(x)$, the MLE of the *pdf* $f(x; \alpha)$ is

$$\hat{f}\left(x; \alpha\right) = \frac{(n+1) f_o\left(x\right)}{U(R_n)\left[1 - F_o\left(x\right)\right]} exp\left(-(n+1)\frac{U(x)}{U(R_n)}\right) \tag{5}$$

The following theorem provides the MLE of $P = P(X < Y)$ when X and Y belong to different family of distributions of PHR model.

Theorem 1. *The MLE of $P(X < Y)$ when X and Y belong to different families of distributions, is*

$$\hat{P} = \int_0^\infty e^{-z} exp\left(\frac{-(m+1)}{V\left(R_m^*\right)} \log\left(\frac{1}{1 - G_o\ F_o^{-1}(1 - e^{\frac{-zU(R_n)}{n+1}})}\right)\right) dz$$

Proof. We have,

$$\hat{P} = \int_{x=0}^\infty \int_{y=x}^\infty \hat{g}\left(y; \beta\right) \hat{f}\left(x; \alpha\right) dx dy$$

$$= \int_{x=0}^\infty \hat{R}\left(x\right) \hat{f}\left(x; \alpha\right) dx$$

$$= \int_{x=0}^\infty \exp\left(\frac{-(m+1) V(x)}{V\left(R_m^*\right)}\right) \left\{\frac{(n+1) f_o(x)}{U(R_n)\left[1 - F_o\left(x\right)\right]}\right\} \exp\left(\frac{-(n+1) U(x)}{U(R_n)}\right) dx$$

The result now follows on substituting $\frac{(n+1)U(x)}{U(R_n)} = z$. $\square$

The following corollary is a particular case of Theorem 1 when X and Y belong to the same family of distributions with *pdf* $f(x; \alpha) = \alpha f_o(x)\left(\overline{F}_o(x)\right)^{\alpha-1}$ and $f(x; \beta) = \beta f_o(x)\left(\overline{F}_o(x)\right)^{\beta-1}$ respectively. The result in Corollary 1 is also a result by Basirat et al. (2016) but by a different approach.

Corollary 1. *When X and Y belong to same family of distributions, the MLE of P is given by*

$$\hat{P} = \frac{U(R_m^*)}{U(R_m^*) + \frac{(m+1)}{(n+1)}U(R_n)}$$

Proof. In Theorem 1, put $G_o = F_o$, hence $U(x) = V(x)$ and the result follows. □

4. UMVUE

Under this section we focus our attention on unbiased estimator of the population parameter as MLE need not be unbiased. Using the sufficiency and completeness criterion, we obtain an unbiased estimator of α that has minimum variance in a class of all unbiased estimators of α. Such an estimator is called a uniformly minimum variance unbiased estimator. Since $U(R_n)$ is distributed as a gamma variate, then on considering $E\left(U(R_n)^{-q}\right) = \left(\frac{\Gamma(n-q+1)}{\Gamma(n+1)}\right)\alpha^q$, we obtain the UMVUE of α^q as

$$\widetilde{\alpha^q} = \begin{cases} \left(\frac{\Gamma(n+1)}{\Gamma(n-q+1)}\right)(U(R_n))^{-q}; & n > q - 1 \\ 0; & \text{otherwise} \end{cases} \tag{6}$$

It is easy to see that $R(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}(\alpha U(t))^i$. Then from (6), we get

$$\tilde{R}(t) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!}\{U(t)\}^i \widetilde{\alpha^i}$$

$$= \sum_{i=0}^{n} (-1)^i \binom{n}{i}\left\{\frac{U(t)}{U(R_n)}\right\}^i$$

which gives the UMVUE of $R(t)$ as

$$\tilde{R}(t) = \begin{cases} \left(1 - \frac{U(t)}{U(R_n)}\right)^n; & U(t) < U(R_n) \\ 0; & \text{otherwise} \end{cases} \tag{7}$$

Taking the derivative of $\tilde{R}(x)$ with respect to x, we get the UMVUE of *pdf* $f(x; \alpha)$ as

$$\tilde{f}(x; \alpha) = \begin{cases} \frac{n f_o(x)}{U(R_n)[1-F_o(x)]}\left[1 - \frac{U(x)}{U(R_n)}\right]^{n-1}; & U(x) < U(R_n) \\ 0; & \text{otherwise} \end{cases} \tag{8}$$

The following theorem provides the UMVUE of P when X and Y belong to different family of distributions of PHR model.

Theorem 2. *The UMVUE of P is given by*

$$\tilde{P} = \begin{cases} n\int_0^{\frac{\log(1/(1-F_oG_o^{-1}(1-e^{-V(R_m^*)})))}{U(R_n)}} (1-z)^{n-1}\left[1 - V(R_m^*)^{-1}\log\left\{\frac{1}{1-G_oF_o^{-1}\left(1-e^{-zU(R_n)}\right)}\right\}\right]^m dz; \\ \qquad\qquad G_o^{-1}\left(1-e^{-V(R_m^*)}\right) < F_o^{-1}\left(1-e^{-U(R_n)}\right) \\ \\ n\int_0^1 (1-z)^{n-1}\left[1 - V(R_m^*)^{-1}\log\left\{\frac{1}{1-G_oF_o^{-1}\left(1-e^{-zU(R_n)}\right)}\right\}\right]^m dz; \\ \qquad\qquad G_o^{-1}\left(1-e^{-V(R_m^*)}\right) \geq F_o^{-1}\left(1-e^{-U(R_n)}\right) \end{cases}$$

Proof. From the arguments similar to those used in the proof of Theorem 1,

$$\tilde{P} = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \tilde{g}\left(y;\beta\right) \tilde{f}\left(x;\alpha\right) dxdy$$

$$= \int_{x=0}^{\infty} \tilde{R}\left(x\right) \tilde{f}\left(x;\alpha\right) dx$$

$$= \int_{0}^{min\left[G_o^{-1}\left(1-e^{-V(R_m^*)}\right),\ F_o^{-1}\left(1-e^{-U(R_n)}\right)\right]} \left(1 - \frac{V\left(x\right)}{V\left(R_m^*\right)}\right)^m \left(\frac{n\ f_o(x)}{U(R_n)\left[1-F_o\left(x\right)\right]}\right)$$

$$\times \left(1 - \frac{U(x)}{U(R_n)}\right)^{n-1} dx$$

The theorem now follows on considering the two cases and substituting $\frac{U(x)}{U(R_n)} = z$.   □

The following corollary is a particular case of Theorem 2 when X and Y belong to the same family of distributions with *pdf* $f\left(x;\alpha\right) = \alpha f_o(x)\left(\overline{F}_o\left(x\right)\right)^{\alpha-1}$ and $f\left(x;\beta\right) = \beta f_o(x)\left(\overline{F}_o\left(x\right)\right)^{\beta-1}$ respectively. The result in Corollary 2 is also a result by Basirat et al. (2016) but by a different approach.

Corollary 2. *When X and Y belong to same family of distributions, the MLE of P is given by*

$$\tilde{P} = \begin{cases} \sum_{i=0}^{m} (-1)^i \frac{\binom{m}{i}}{\binom{n+i}{i}} \left\{\frac{U\left(R_n\right)}{U\left(R_m^*\right)}\right\}^i; & U(R_n) \le U(R_m^*) \\ 1 - \sum_{i=0}^{n} (-1)^i \frac{\binom{n}{i}}{\binom{m+i}{i}} \left\{\frac{U\left(R_m^*\right)}{U\left(R_n\right)}\right\}^i; & U\left(R_n\right) > U(R_m^*) \end{cases}$$

Proof. In Theorem 2, put $G_o = F_o$, hence $U\left(x\right) = V\left(x\right)$ and we get

$$\tilde{P} = \begin{cases} n \int_{z=0}^{1} (1-z)^{n-1} \left(1 - z\frac{U\left(R_n\right)}{U\left(R_m^*\right)}\right)^m dz; & U(R_n) \le U(R_m^*) \\ n \int_{z=0}^{\frac{U(R_m^*)}{U(R_n)}} (1-z)^{n-1} \left(1 - z\frac{U\left(R_n\right)}{U\left(R_m^*\right)}\right)^m dz; & U\left(R_n\right) > U(R_m^*) \end{cases}$$

For $U(R_n) \le U(R_m^*)$, expand $\left(1 - z\frac{U\left(R_n\right)}{U(R_m^*)}\right)^m$ binomially to obtain the result and for $U\left(R_n\right) > U(R_m^*)$, put $z\frac{U\left(R_n\right)}{U(R_m^*)} = w$ to obtain

$$\tilde{P} = n\frac{U\left(R_m^*\right)}{U\left(R_n\right)} \int_{w=0}^{1} \left(1 - \frac{U\left(R_m^*\right)}{U\left(R_n\right)}w\right)^{n-1} (1-w)^m dw$$

On integrating by parts and then expanding $\left(1 - \frac{U(R_m^*)}{U(R_n)}w\right)^n$ binomially we obtain the result.
          □

5. Bayesian estimators

So far we have considered parameter α as a fixed constant. In Bayesian estimation theory, we treat α as a random variable having a priori distribution. Another element of this theory is the specification of a loss function which measures the loss incurred while taking a decision on the basis of a sample drawn from the population. While adopting Bayesian estimation procedures, one has to assume the prior distribution of the parameters and the loss function.

Let us assume that the natural conjugate prior distribution of parameter α of PHR model is Gamma(ν, μ) with *pdf*

$$\pi(\alpha) = \frac{\mu^\nu}{\Gamma(\nu)}\alpha^{\nu-1}e^{-\mu\alpha}; \ \alpha \geq 0, \ \mu \geq 0 \text{ and } \nu \text{ is a positive integer} \tag{9}$$

Combining (2) and (9) via Bayes' theorem, the posterior density of α comes out to be Gamma$(n + \nu + 1, U(R_n) + \mu)$ with *pdf*

$$\pi^*(\alpha \mid U(R_n)) = \frac{(U(R_n) + \mu)^{n+\nu+1}}{\Gamma(n+\nu+1)}\alpha^{n+\nu}e^{-\alpha(U(R_n)+\mu)} \tag{10}$$

Using the fact that $U(R_n)$ follows Gamma$(n + 1, \alpha)$, then from (9), the marginal *pdf* of $U(R_n)$ is

$$\varphi(U(R_n)) = \int_{\alpha=0}^{\infty} \frac{\alpha^{n+1}}{\Gamma(n+1)}(U(R_n))^n e^{-\alpha U(R_n)}\pi(\alpha)\,d\alpha$$

$$= \frac{\mu^\nu(U(R_n))^n}{B(n+1,\nu)(U(R_n)+\mu)^{n+\nu+1}} \ ; \ U(R_n) > 0 \tag{11}$$

where $B(r, s)$ is beta function. Under squared error loss function (SELF), the Bayes estimator of α^q denoted by $\hat{\alpha}_{BS}$, is the posterior mean. Thus from (10) we get

$$\hat{\alpha}_{BS} = \frac{\Gamma(n+\nu+q+1)}{\Gamma(n+\nu+1)}(U(R_n)+\mu)^{-q} \tag{12}$$

We can re-write the *pdf* of PHR model as $f(x;\alpha) = \frac{f_o(x)}{1-F_o(x)}\sum_{i=0}^{\infty}\frac{(-1)^i}{i!}(U(x))^i\alpha^{i+1}$. Then on using Lemma 1 of Chaturvedi and Tomer (2002) and (12) we get the Bayes estimator of the *pdf* $f(x;\alpha)$ as

$$\hat{f}_{BS}(x;\alpha) = \frac{f_o(x)(n+\nu+1)}{(1-F_o(x))(U(R_n)+\mu)}\left(1 + \frac{U(x)}{U(R_n)+\mu}\right)^{-(n+\nu+2)} \tag{13}$$

Next, we obtain the Bayes estimator of $R(t)$ denoted by $\hat{R}(t)_{BS}$ on considering $\hat{R}(t)_{BS} = \int_t^{\infty}\hat{f}_{BS}(x;\alpha)\,dx$ which gives

$$\hat{R}(t)_{BS} = \left(1 + \frac{U(t)}{U(R_n)+\mu}\right)^{-(n+\nu+1)} \tag{14}$$

on substituting $\frac{U(x)}{U(R_n)+\mu} = z$.

Finally, for obtaining Bayes estimator of P, let X and Y be two independent random variables from different family of distributions of PHR model with *pdf* $f(x;\alpha)$ and $g(y;\beta)$ as previously defined. Further, let us assume that the prior densities of α and β are Gamma(ν_1, μ_1) and Gamma(ν_2, μ_2) respectively. Then the following theorem provides the Bayes estimator of $P(X < Y)$ denoted by $\hat{P}_{BS}$.

Theorem 3. *The Bayes estimator of $P(X < Y)$ when X and Y belong to different family of distributions of PHR model, is*

$$\hat{P}_{BS} = \left(\frac{n+\nu_1+1}{U(R_n)+\mu_1}\right)\int_{w=0}^{\infty}\left(1 + \frac{w}{U(R_n)+\mu_1}\right)^{-(n+\nu_1+2)}$$

$$\times \left[1 + \frac{\log\left(\frac{1}{1-G_o F_o^{-1}(1-e^{-w})}\right)}{V(R_m^*)+\mu_2}\right]^{-(m+\nu_2+1)} dw$$

Proof.

$$\hat{P}_{BS} = \int_{x=0}^{\infty} \int_{y=x}^{\infty} \hat{g}_{BS}(y;\beta)\hat{f}_{BS}(x;\alpha)\,dxdy$$

$$= \int_{x=0}^{\infty} \hat{R}(x)_{BS}\hat{f}_{BS}(x;\alpha)\,dx$$

$$= \int_{x=0}^{\infty} \left(1 + \frac{V(x)}{V(R_m^*) + \mu_2}\right)^{-(m+\nu_2+1)} \left\{\frac{(n+\nu_1+1)f_o(x)}{(U(R_n)+\mu_1)(1-F_o(x))}\right\}$$

$$\times \left(1 + \frac{U(x)}{U(R_n)+\mu_1}\right)^{-(n+\nu_1+2)} dx$$

The result now follows on substituting $\frac{U(x)}{U(R_n)+\mu_1} = z$.

The following corollary is a particular case of Theorem 3 when X and Y belong to the same family of distributions with *pdf* $f(x;\alpha) = \alpha f_o(x)\left(\overline{F}_o(x)\right)^{\alpha-1}$ and $f(x;\beta) = \beta f_o(x)\left(\overline{F}_o(x)\right)^{\beta-1}$ respectively. The result in Corollary 3 is also a result by Basirat et al. (2016) but by a different approach.

Corollary 3. *The Bayes estimator of $P(X < Y)$ when X and Y belong to the same family of distributions of PHR model, is*

$$\hat{P}_{BS} = \left(\frac{n+\nu_1+1}{n+m+\nu_1+\nu_2+2}\right)\left(\frac{U(R_n)+\mu_1}{U(R_m^*)+\mu_2}\right)^{n+\nu_1+1}$$

$$\times F_{2,1}\left(n+m+\nu_1+\nu_2+2, n+\nu_1+2; n+m+\nu_1+\nu_2+3, 1 - \frac{U(R_n)+\mu_1}{U(R_m^*)+\mu_2}\right)$$

Proof. In Theorem 3, put $G_o = F_o$, hence $U(x) = V(x)$ and we get

$$\hat{P}_{BS} = \left(\frac{n+\nu_1+1}{U(R_n)+\mu_1}\right)\int_{w=0}^{\infty} \left(1 + \frac{w}{U(R_n)+\mu_1}\right)^{-(n+\nu_1+2)}\left[1 + \frac{w}{U(R_m^*)+\mu_2}\right]^{-(m+\nu_2+1)} dw$$

Substitute $1 + \frac{w}{U(R_m^*)+\mu_2} = \frac{1}{z}$ and we obtain for $k = 1 - \frac{U(R_n)+\mu_1}{U(R_m^*)+\mu_2}$,

$$\hat{P}_{BS} = \left(\frac{n+\nu_1+1}{1-k}\right)\int_0^1 \frac{z^{m+\nu_2-1}}{\left[1 + \frac{1-z}{z(1-k)}\right]^{n+\nu_1+2}} dz$$

$$= (n+\nu_1+1)(1-k)^{n+\nu_1+1}\int_0^1 z^{n+m+\nu_1+\nu_2+1}(1-kz)^{-(n+\nu_1+2)} dz$$

The result follows on using the result of Gradshteyn and Ryzhik, section 3.197.3 given by

$$F_{2,1}(b,a;c,x) = \frac{1}{B(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tx)^{-a}dt; |c| < 1$$

where $B(r,s)$ is beta function.

6. Confidence intervals

For constructing confidence intervals of the parameters of the PHR model, we first obtain the Fisher information of α as:

$$I(\alpha) = -E\left[\frac{\partial^2 \log(L(\alpha|R_0,R_1,R_2,\ldots,R_n))}{\partial\alpha^2}\right] \text{ where } \frac{\partial^2 \log(L(\alpha|R_0,R_1,R_2,\ldots,R_n))}{\partial\alpha^2} = \frac{-(n+1)}{\alpha^2}.$$

Since it is a complicated task to obtain the expectation of the above expressions, therefore we use observed Fisher information of α, which is obtained by dropping the expectation

sign. The asymptotic variance of the MLE is the inverse of $I(\widehat{\alpha}) = \frac{-(n+1)}{\alpha^2}$. After obtaining the inverse, we get variance of $\widehat{\alpha}$. We use this value to construct confidence intervals of α. Assuming asymptotic normality of the MLE, CI for α is constructed. Let $\widehat{\sigma}^2(\widehat{\alpha})$ be the estimated variance of $\widehat{\alpha}$. Then $100(1-\varepsilon)\%$ asymptotic CI for α is given by:

$$\left(\widehat{\alpha} - Z_{\frac{\varepsilon}{2}}\widehat{\sigma}(\widehat{\alpha}), \widehat{\alpha} + Z_{\frac{\varepsilon}{2}}\widehat{\sigma}(\widehat{\alpha})\right)$$

where $Z_{\frac{\varepsilon}{2}}$ is the upper $100(1-\varepsilon)$ percentile point of standard normal distribution and $\widehat{\sigma}(\widehat{\alpha})$ is the estimated standard deviation of $\widehat{\alpha}$. Using this CI, one can easily construct $100(1-\varepsilon)\%$ CI for $R(t)$ as follows:

$$\left(\exp(-U(t)(\widehat{\alpha} + Z_{\frac{\varepsilon}{2}}\widehat{\sigma}(\widehat{\alpha}))), \exp(-U(t)(\widehat{\alpha} - Z_{\frac{\varepsilon}{2}}\widehat{\sigma}(\widehat{\alpha})))\right).$$

Meeker and Escober (1998) reported that the asymptotic CI based on log transformation of MLE has better coverage probability. An approximate $100(1-\varepsilon)\%$ CI for $\log(\alpha)$ is:

$$\left(\log(\widehat{\alpha}) - Z_{\frac{\varepsilon}{2}}\widehat{\sigma}(\log(\widehat{\alpha})), \log(\widehat{\alpha}) + Z_{\frac{\varepsilon}{2}}\widehat{\sigma}(\log(\widehat{\alpha}))\right)$$

where $\widehat{\sigma}^2(\log(\widehat{\alpha}))$ is the estimated variance of $\log(\alpha)$ and is approximated by $\widehat{\sigma}^2(\log(\widehat{\alpha})) = \frac{\widehat{\sigma}^2(\widehat{\alpha})}{\widehat{\alpha}^2}$. Hence, approximate $100(1-\varepsilon)\%$ CI for α is:

$$\left(\widehat{\alpha}e^{-Z_{\frac{\varepsilon}{2}}\frac{\widehat{\sigma}(\widehat{\alpha})}{\widehat{\alpha}}}, \widehat{\alpha}e^{Z_{\frac{\varepsilon}{2}}\frac{\widehat{\sigma}(\widehat{\alpha})}{\widehat{\alpha}}}\right).$$

Next, we consider two independent $rvs\, X$ and Y belonging to the PHR model with pdf $f(x;\alpha) = \alpha f_o(x)(\overline{F}_o(x))^{\alpha-1}$ and $g(y;\beta) = \beta g_o(y)(\overline{G}_o(y))^{\beta-1}$ respectively as defined earlier. We develop confidence interval for the ratio of the parameters $\mu = \frac{\alpha}{\beta}$ using the fact that $2\alpha U(R_n) \sim \chi^2_{2(n+1)}$ and $2\beta V(R_m^*) \sim \chi^2_{2(m+1)}$. Thus, we get $F = \frac{2\alpha U(R_n)/2(n+1)}{2\beta V(R_m^*)/2(m+1)} \sim F_{2(n+1),2(m+1)}$ and $100(1-\varepsilon)\%$ CI of μ is:

$$\left(\frac{(n+1)V(R_m^*))}{(m+1)U(R_n)}F_{2(n+1),2(m+1)}\left(1 - \frac{\varepsilon}{2}\right), \frac{(n+1)V(R_m^*)}{(m+1)U(R_n)}F_{(2(n+1),2(m+1)}\left(\frac{\varepsilon}{2}\right)\right)$$

where $F_{2(n+1),2(m+1)}(\delta)$ is the δth percentile of F-distribution with $2(n+1)$ and $2(m+1)$ degrees of freedom.

Now, if $rvs\, X$ and Y belong to the same family of distributions from the PHR model with common shape parameter α, i.e. $X \sim f(x;\alpha)$ and $Y \sim f(y;\alpha)$, then the joint distribution of $n+1$ records from the istributin of X and $m+1$ records from the distribution of Y is:

$$L(\alpha|R_o, R_1, \ldots, R_n, R_o^*, R_1^*, \ldots, R_m^*)$$

$$= \alpha^{n+m+2}\exp(-\alpha(U(R_n) + U(R_m^*)))\prod_{i=0}^{n}\frac{f_o(R_i)}{1 - F_o(R_i)}\prod_{i=0}^{m}\frac{f_o(R_i^*)}{1 - F_o(R_i^*)}.$$

Thus the MLE of α is $\frac{n+m+2}{U(R_n)+U(R_m^*)}$. Using the fact that $2\alpha(U(R_n) + U(R_m^*)) \sim \chi^2_{2(n+m+2)}$, the $100(1-\varepsilon)\%$ CI of α is:

$$\left(\frac{\chi^2_{2(n+m+2)}\left(1 - \frac{\varepsilon}{2}\right)}{2(U(R_n) + U(R_m^*))}, \frac{\chi^2_{2(n+m+2)}\left(\frac{\varepsilon}{2}\right)}{2(U(R_n) + U(R_m^*))}\right)$$

Finally, in order to construct confidence interval of P, we consider the case when $rvs\, X$ and Y belong to the same family of distributions from the PHR model with parameters α and β respectively. Since the pdf of $U(R_n)$ is $Gamma(n+1, \alpha)$, thus from (3), the pdf of MLE of α is inverted-$Gamma(n+1, (n+1)\alpha)$. Similarly, the pdf of MLE of β is inverted-$Gamma(m+1, (m+1)\beta)$. Then we get $W_x = \frac{2(n+1)\alpha}{\widehat{\alpha}} \sim \chi^2_{2(n+1)}$ and $W_y = \frac{2(m+1)\beta}{\widehat{\beta}} \sim \chi^2_{2(m+1)}$

where $\widehat{\alpha}$ is defined in (3) and $\hat{\beta} = \frac{m+1}{U(R_m^*)}$. Thus, $Z = \frac{W_x/2(n+1)}{W_y/2(m+1)} \sim F_{2(n+1),2(m+1)}$ which gives $\frac{\beta}{\alpha} \sim \frac{\hat{\beta}}{\widehat{\alpha} F_{2(n+1),2(m+1)}}$ and we get $100(1-\varepsilon)\%$ CI of $P = \frac{1}{1+\frac{\beta}{\alpha}}$ as:

$$\left(\left(1 + \frac{\hat{\beta}}{\widehat{\alpha} F_{2(n+1),2(m+1)} \left(1 - \frac{\varepsilon}{2}\right)} \right)^{-1}, \left(1 + \frac{\hat{\beta}}{\widehat{\alpha} F_{2(n+1),2(m+1)} \left(\frac{\varepsilon}{2}\right)} \right)^{-1} \right)$$

7. Hypotheses testing

Suppose we have to test the hypothesis $H_o : \alpha = \alpha_o$ against $H_1 : \alpha \neq \alpha_o$. It follows from (2) that, under H_o,

$$\sup_{\Theta_o} L(\alpha|R_o, R_1, \ldots, R_n) = (\alpha_o)^{n+1} \exp(-\alpha_0 U(R_n)) \prod_{i=o}^{n} \frac{f_o(R_i)}{1 - F_o(R_i)}; \ \theta_o = \{\alpha : \alpha = \alpha_0\}$$

and

$$\sup_{\Theta} L(\alpha|R_0, R_1, \ldots, R_n) = \left(\frac{n+1}{U(R_n)} \right)^{n+1} \exp(-(n+1)) \prod_{i=0}^{n} \frac{f_o(R_i)}{1 - F_o(R_i)}; \ \Theta = \{\alpha : \alpha > 0\}$$

The likelihood ratio is given by:

$$\Phi(R_0, R_1, \ldots, R_n) = \frac{\sup_{\Theta_o} L(\alpha|R_0, R_1, \ldots, R_n)}{\sup_{\Theta} L(\alpha|R_0, R_1, \ldots, R_n)}$$

$$= \left\{ \frac{\alpha_o U(R_n)}{(n+1)} \right\}^{n+1} \exp\{-\alpha_o U(R_n) + (n+1)\} \tag{15}$$

We note that the first term on the right hand side of (15) is monotonically increasing and the second term is monotonically decreasing in $U(R_n)$. Since $2\alpha_o U(R_n) \sim \chi^2_{2(n+1)}$, the critical region is given by:

$$\{0 < U(R_n) < k_o\} \cup \{k'_o < U(R_n) < \infty\}$$

where k_o and k'_o are obtained such that $k_o = \frac{\chi^2_{2(n+1)}\left(\frac{\varepsilon}{2}\right)}{2\alpha_o}$ and $k'_o = \frac{\chi^2_{2(n+1)}\left(1 - \frac{\varepsilon}{2}\right)}{2\alpha_o}$ where ε is the level of significance.

An important hypothesis in life-testing experiments is $H_o : \alpha \leq \alpha_o$ against $H_1 : \alpha > \alpha_o$. It follows from (2) that for $\alpha_1 > \alpha_2$,

$$\frac{L(\alpha_1|R_0, R_1, \ldots, R_n)}{L(\alpha_2|R_0, R_1, \ldots, R_n)} = \left(\frac{\alpha_1}{\alpha_2} \right)^{n+1} \exp\{(\alpha_2 - \alpha_1)U(R_n)\} \tag{16}$$

It follows from (16) that the family of distributions $f(x; \alpha)$ has monotone likelihood ratio in $U(R_n)$. Thus, the uniformly most powerful critical region for testing H_o against H_1 is given by [see Lehmann (1959, p.88)]

$$\Phi(R_0, R_1, \ldots, R_n) = \begin{cases} 1; & U(R_n) \leq k''_o \\ 0; & \text{otherwise} \end{cases}$$

where $k''_o = \frac{\chi^2_{2(n+1)}(\varepsilon)}{2\alpha_o}$.

It can be seen that when X and Y belong to same families of distributions, then $P = \frac{1}{1+\frac{\beta}{\alpha}}$. Suppose we want to test $H_o : P = P_o$ against $H_1 : P \neq P_o$. It follows that H_o is equivalent to $\alpha = k\beta$ where $k = \frac{P_o}{1-P_o}$. Thus, $H_o : \alpha = k\beta$ and $H_1 : \alpha \neq k\beta$.

The likelihood for observing α and β is

$$L(\alpha, \beta | R_0, R_1, \ldots, R_n, R_0^*, R_1^*, \ldots, R_m^*)$$

$$= (\alpha)^{n+1}(\beta)^{m+1} \exp(-\{\alpha U(R_n) + \beta U(R_m^*)\}) \prod_{i=0}^{n} \frac{f_o(R_i)}{1 - F_o(R_i)} \prod_{i=0}^{m} \frac{f_o(R_i^*)}{1 - F_o(R_i^*)}$$

It can be shown that, under H_o,

$$\hat{\beta} = \frac{n+m+2}{kU(R_n) + U(R_m^*)} \quad \text{and} \quad \hat{\alpha} = \frac{k(n+m+2)}{kU(R_n) + U(R_m^*)}$$

Thus, for a generic constant C,

$$\sup_{\Theta_o} L(\alpha, \beta | R_0, R_1, \ldots, R_n, R_0^*, R_1^*, \ldots, R_m^*)$$

$$= \frac{C}{\{kU(R_n) + U(R_m^*)\}^{n+m+2}} \exp\{-(n+m+2)\}; \quad \Theta_0 = \{\alpha, \beta : \alpha = k\beta\} \qquad (17)$$

and

$$\sup_{\Theta} L(\alpha_1, \alpha_2 | R_0, R_1, \ldots, R_n, R_0^*, R_1^*, \ldots, R_m^*)$$

$$= \frac{C}{\{U(R_n)\}^{n+1}\{U(R_m^*)\}^{m+1}} \exp\{-(n+m+2)\}; \quad \Theta = \{\alpha_1, \alpha_2 : \alpha_1 > 0, \alpha_2 > 0\} \qquad (18)$$

From (17) and (18), the likelihood ratio is:

$$\Phi(R_0, R_1, \ldots, R_n, R_0^*, R_1^*, \ldots, R_m^*) = \frac{C\left\{\frac{U(R_n)}{U(R_m^*)}\right\}^{n+1}}{\left\{1 + \frac{kU(R_n)}{U(R_m^*)}\right\}^{n+m+2}}$$

Using the fact that $\frac{U(R_n)}{U(R_m^*)} \sim \frac{(n+1)\beta}{(m+1)\alpha} F_{2(n+1),2(m+1)}$, the critical region is given by

$$\left\{\frac{U(R_n)}{U(R_m^*)} < k_2\right\} \cup \left\{\frac{U(R_n)}{U(R_m^*)} > k_2'\right\}$$

where $k_2 = \frac{(n+1)}{k(m+1)} F_{2(n+1),2(m+1)}\left(\frac{\varepsilon}{2}\right)$ and $k_2' = \frac{(n+1)}{k(m+1)} F_{2(n+1),2(m+1)}\left(1 - \frac{\varepsilon}{2}\right)$.

8. Real data example

This section deals with an example of real data to illustrate the proposed estimation methods. Let us assign the random variable $X \sim f(x; \alpha)$ to the first data set that has been taken from Lawless (2003, p. 574) and has been reproduced in the following table.

Data Set I

184	241	273	1842	371	830	683	1306	562	166
981	1867	493	418	2978	1463	2220	312	251	76

Basirat et al. (2016) used Kolmogorov-Smirnov (K-S) test and concluded that exponential distribution is adequate for the above data set. The following figure also confirms the same for $\alpha = 0.001141$.
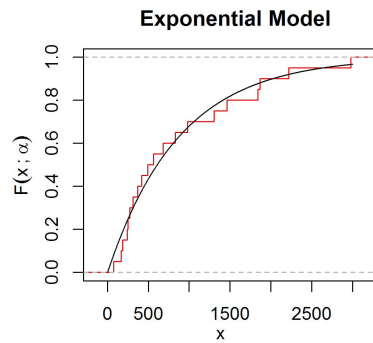
Figure 2: The empirical and theoretical *cdf* of Exponential(α) model.

Now let us assign the random variable $Y \sim g(y; \beta)$ to the second data set that has been taken from Lawless (1982, p. 185) concerning the time to breakdown of an insulating fluid between electrodes at a voltage of 34 kV (minutes). The 19 times to breakdown are

Data Set II

0.96	4.15	0.19	0.78	8.01	31.75	7.35	6.50	8.27	33.91
32.52	3.16	4.85	2.78	4.67	1.31	12.06	36.71	72.89	

Chaturvedi and Malhotra (2016) used K-S test and concluded that Weibull distribution is adequate for this data with parameters $p = 0.7708$ and $\beta = 0.1452$. The following figure also confirms the same.
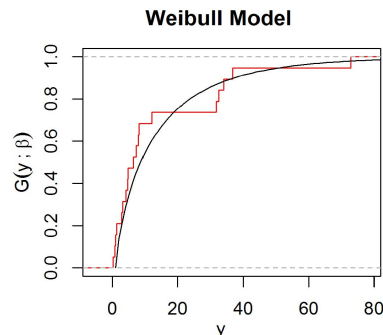

Figure 3: The empirical and theoretical *cdf* of Weibull(p, β) model.

The upper record values based on data set I are 184, 241, 273, 1842, 1867, 2978 and based on data set II are 0.96, 4.15, 8.01, 31.75, 33.91, 36.71, 72.89. Thus, $(n, m) = (5, 6)$, $U(R_n) = 2978$ and $V(R_m^*) = 27.2762$. The following Table 1 shows the different estimators of parameter α of exponential model and its corresponding reliability function $R(t)$ based on data set I. Similarly Table 2 shows the different estimators of parameters p and β of Weibull distribution and its corresponding reliability function $R(t)$ based on data set II.

Table 1: The MLE, UMVUE and Bayes estimator of parameter α of exponential model and its corresponding reliability function $R(t)$ for time $t = 2$ based on data set I.

$\widehat{\alpha}$	$\widetilde{\alpha}$	$\widehat{\alpha}_{BS}$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)_{BS}$
0.00114	0.00168	0.00235	0.99772	0.99665	0.99531

Table 2: The MLE, UMVUE and Bayes estimator of parameters p and β of Weibull distribution and its corresponding reliability function $R(t)$ for time $t = 2$ based on data set II.

$\hat{p}$	$\widehat{\beta}$	$\widetilde{\beta}$	$\widehat{\beta}_{BS}$	$\hat{R}(t)$	$\tilde{R}(t)$	$\hat{R}(t)_{BS}$
0.77082	0.14521	0.21997	0.28292	0.78054	0.67870	0.62580

Now, for the above two data sets belonging to different family of distributions, we obtain estimators of $P = P(X < Y)$ and the results are presented in Table 3. For Bayes estimator of P we assume the prior distribution of α and β to be Gamma(2,2).

Table 3: The MLE, UMVUE and Bayes estimator of $P(X < Y)$.

$\hat{P}$	$\tilde{P}$	$\hat{P}_{BS}$
0.01345	0.01101	0.01671

9. Numerical computations and simulations

In this section, we numerically analyse the different estimators of parametric functions derived in the preceding sections along with their bias and mean square error (MSE). First we study the behaviour of estimators of parameter α of the PHR model. Note that to obtain the bias and MSE of estimators of α, we do not require to assume any baseline distribution function F_o. Thus the results obtained will in general hold for all the distributions belonging to the PHR model. The following Figure 4 compares the MSE of MLE, UMVUE and Bayes estimator of α for a particular sample size n. We observe from this figure that the Bayes estimator of α has a consistently low MSE for several values of α although it seems to be more biased compared to the MLE and UMVUE. As the values of α increases, the UMVUE performs better. This trend is same for small ($n = 5$) and large ($n = 15$) samples. From Figure 5 we observe that all the estimators of α perform better in the sense that they have a smaller bias and MSE with an increase in number of records n.
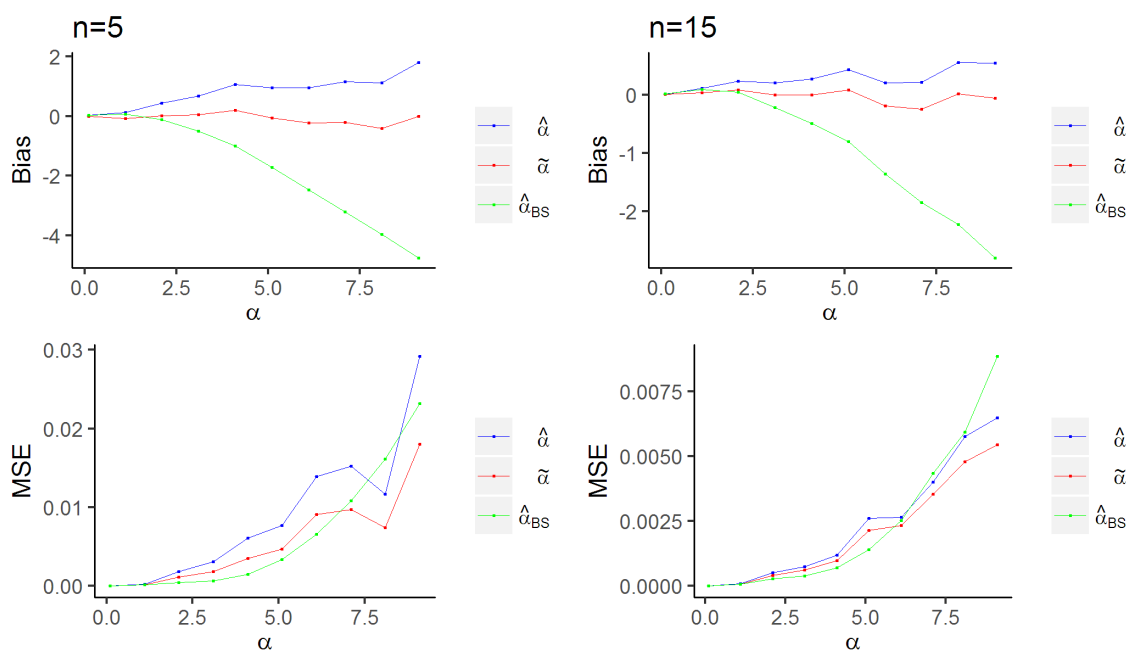

Figure 4: Bias and MSE of MLE, UMVUE and Bayes estimator of α for a specific number of records.
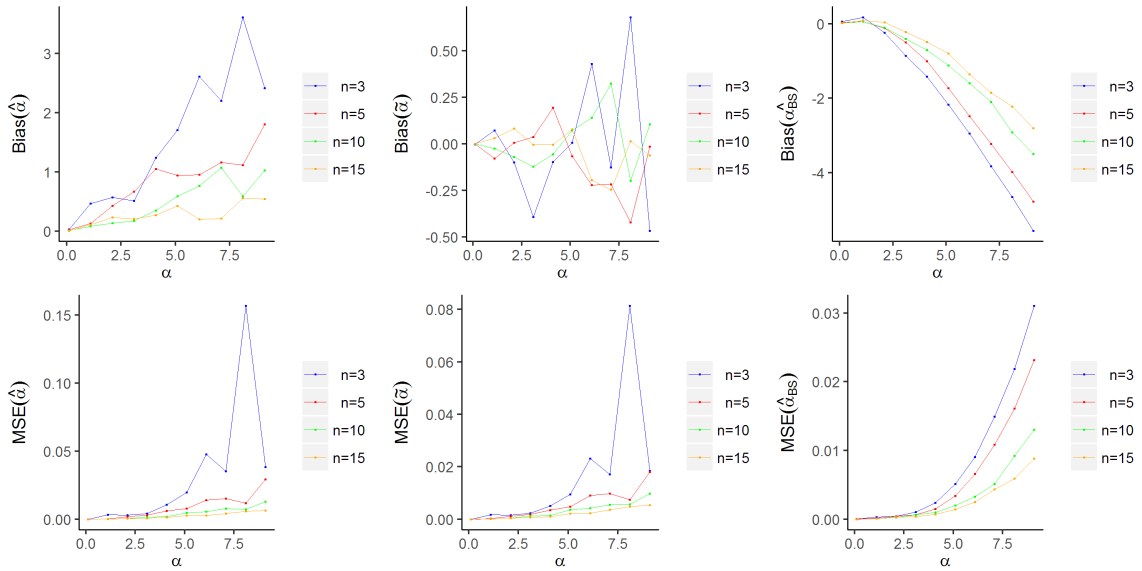
Figure 5: Bias and MSE of various estimators of α for various sample sizes n.

Next, we study the behaviour of estimators of reliability function $R(t)$. We know that as time t increases, the reliability $R(t)$ decreases. We observe from Figure 6 that the MLE and UMVUE of $R(t)$ have a smaller bias and MSE than the Bayes estimator as time t increases and are thus better estimators of $R(t)$. However, for small values of t, all the estimators perform equally well. Figure 7 shows that all the estimators of $R(t)$ perform better in the sense that they have a smaller bias and MSE with an increase in number of records n.
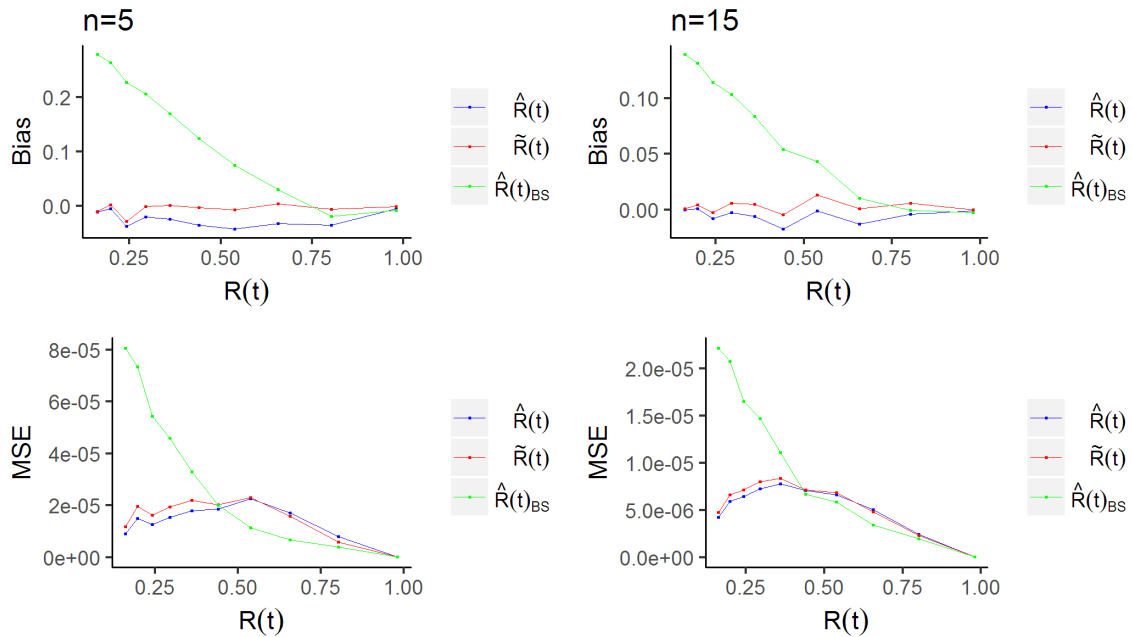

Figure 6: Bias and MSE of MLE, UMVUE and Bayes estimator of $R(t)$ for a specific number of records.
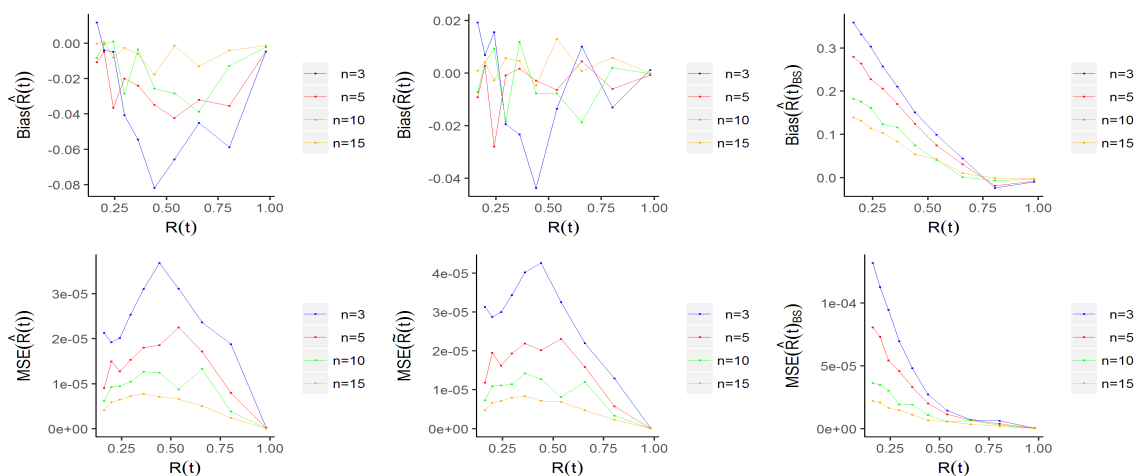
Figure 7: Bias and MSE of various estimators of $R(t)$ for various sample sizes n.

Finally, we discuss the behaviour of estimators of $P(X < Y)$ where X and Y are random variables from different families of PHR model. Let X belong to exponential distribution with parameter α and Y belong to Weibull distribution with parameters p and β. Then Table 4 gives the MSE of MLE, UMVUE and Bayes estimator of P for different values of the parameters and sample sizes. We observe from Table 4 and Figure 8 that the UMVUE of P is a better estimator when P is close to 0 or 1. However, when P is around 0.5, the Bayes estimator of P performs the best followed by the MLE and UMVUE. Basirat et al. (2016) also showed that the UMVUE of P is better than the MLE around the tails while the MLE is better than the UMVUE when P is around 0.5. Also, they produced these results for the case when X and Y belong to the same distribution. On the other hand, we have presented the results for a more general case and also compared the Bayes estimator of P with its MLE and UMVUE. Figure 9 shows that all the estimators of P perform better with an increase in number of records.

Table 4:   MSE of MLE, UMVUE and Bayes estimator of P.

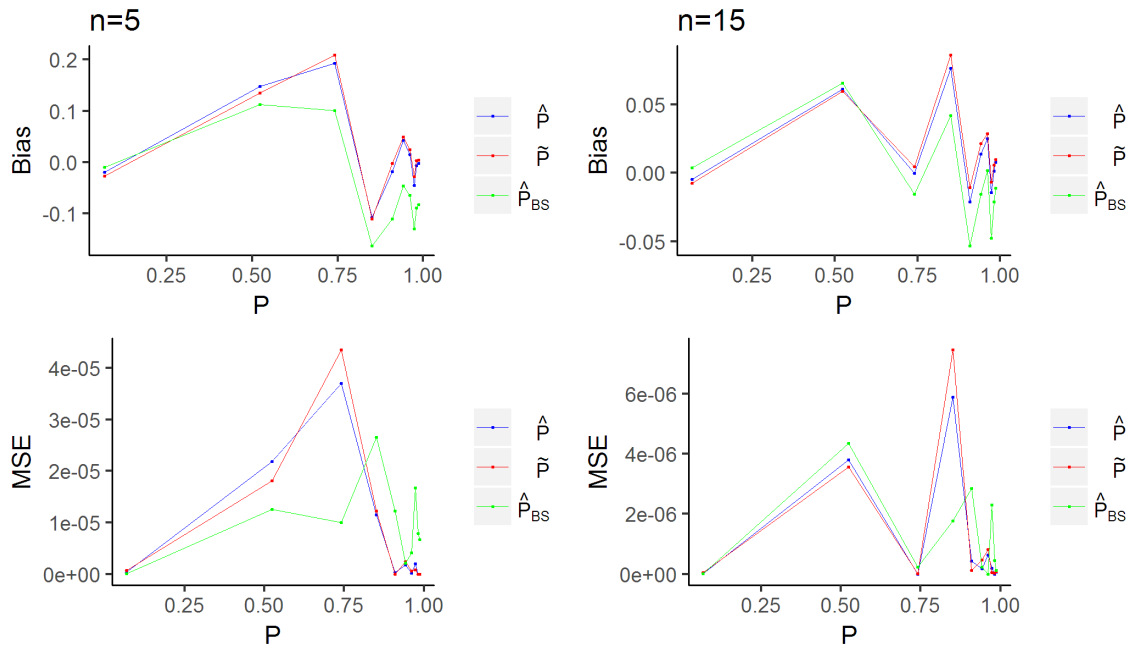| α	(p,β)	P	(n,m)	$\hat{P}$	$MSE(\hat{P})$	$\tilde{P}$	$MSE(\tilde{P})$	$\hat{P}_{BS}$	$MSE(\hat{P}_{BS})$
0.1	(2,3)	0.05	(3,6)	0.06512	0.00219	0.05135	0.00138	0.08858	0.00410
			(5,5)	0.05663	0.00102	0.04899	0.00078	0.07621	0.00205
			(6,3)	0.05628	0.00079	0.05046	0.00063	0.08015	0.00204
0.1	(3,2)	0.07	(3,6)	0.08326	0.00315	0.06577	0.00213	0.10379	0.00466
			(5,5)	0.07835	0.00162	0.06779	0.00120	0.09509	0.00260
			(6,3)	0.07613	0.00116	0.06813	0.00091	0.09457	0.00209
0.5	(2,3)	0.22	(3,6)	0.25370	0.01668	0.21682	0.01473	0.28952	0.01612
			(5,5)	0.24117	0.01085	0.21922	0.00998	0.28399	0.01289
			(6,3)	0.23564	0.00941	0.22034	0.00897	0.29391	0.01353
0.5	(3,2)	0.29	(3,6)	0.33195	0.02257	0.28774	0.02175	0.35216	0.01764
			(5,5)	0.31860	0.01430	0.29143	0.01355	0.34330	0.01335
			(6,3)	0.31563	0.01328	0.29605	0.01292	0.34955	0.01337
2	(2,3)	0.59	(3,6)	0.61611	0.02630	0.59478	0.03304	0.56114	0.01087
			(5,5)	0.59590	0.02025	0.58577	0.02407	0.58096	0.00967
			(6,3)	0.58928	0.02123	0.58901	0.02499	0.60676	0.00928
2	(3,2)	0.72	(3,6)	0.72975	0.02251	0.71722	0.03076	0.65020	0.01600
			(5,5)	0.72903	0.01528	0.72455	0.01891	0.67910	0.01047
			(6,3)	0.71884	0.01615	0.72051	0.01924	0.69105	0.00992
6	(2,3)	0.88	(3,6)	0.87540	0.00802	0.88569	0.00903	0.71647	0.03111
			(5,5)	0.87530	0.00618	0.88731	0.00649	0.77075	0.01561
			(6,3)	0.86503	0.00719	0.88301	0.00704	0.79537	0.01034
6	(3,2)	0.96	(3,6)	0.94867	0.00284	0.96098	0.00250	0.80917	0.02523
			(5,5)	0.94851	0.00243	0.95896	0.00213	0.85518	0.01328
			(6,3)	0.94976	0.00213	0.96181	0.00173	0.87639	0.00882
10	(2,3)	0.95	(3,6)	0.93962	0.00312	0.94969	0.00289	0.75702	0.03882
			(5,5)	0.93832	0.00228	0.94858	0.00192	0.81670	0.01917
			(6,3)	0.93567	0.00266	0.94940	0.00206	0.84595	0.01209
10	(3,2)	0.99	(3,6)	0.98205	0.00059	0.98896	0.00032	0.84631	0.02169
			(5,5)	0.98469	0.00033	0.98994	0.00019	0.89828	0.00910
			(6,3)	0.98245	0.00046	0.98868	0.00026	0.91379	0.00646

Figure 8: Bias and MSE of MLE, UMVUE and Bayes estimator of P for a specific number of records.
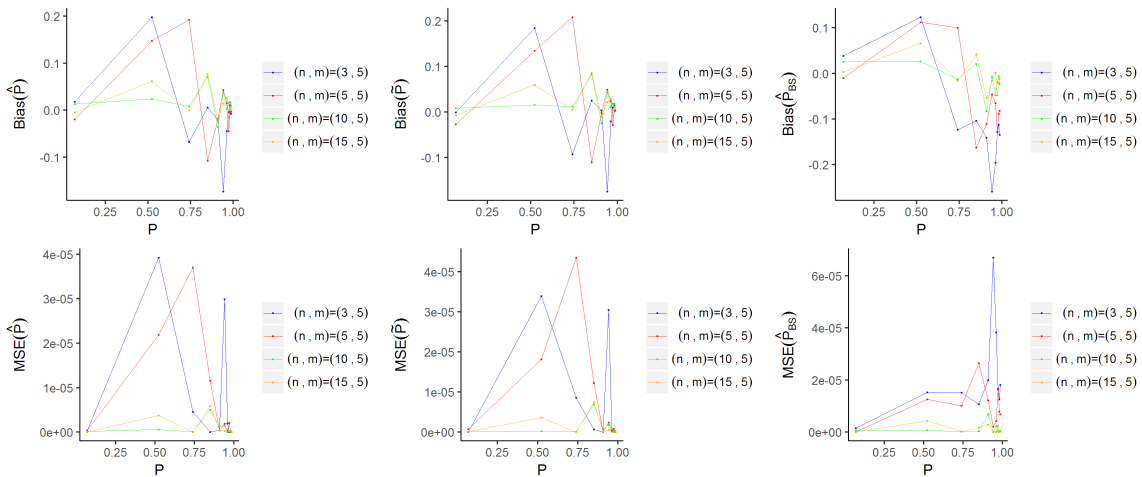

Figure 9: Bias and MSE of various estimators of P for various sample sizes.

10. Discussion

In this paper, we have developed three estimators viz. MLE, UMVUE and Bayes estimator of the parameter and reliability functions $R(t)$ and $P(X < Y)$ of proportional hazard rate model when the data is in the form of record values. We have derived the expressions for $P(X < Y)$ when independent random variables X and Y belong to different families of PHR model, whereas Basirat *et al.* (2016) considered the case when X and Y belong to the same family of PHR model. Moreover, the results have been derived using simpler techniques. Thus, our approach is not only uncomplicated, but also provides more generalised results and one can consider applying our methods when estimating $P(X < Y)$. We have also discussed the behaviour of the developed estimators through simulation studies. Real data sets have also been considered for illustrative purposes. In addition to this we have derived exact and asymptotic confidence intervals for the parameter and reliability functions and have also tested

various hypotheses.

A problem that still remains unsolved is when some prior information exists on the parameter of the PHR model and this information must be incorporated in the model as a constraint, giving rise to restricted model. Since this theory needs further consideration, we leave the idea for future research.

Acknowledgement

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